1.

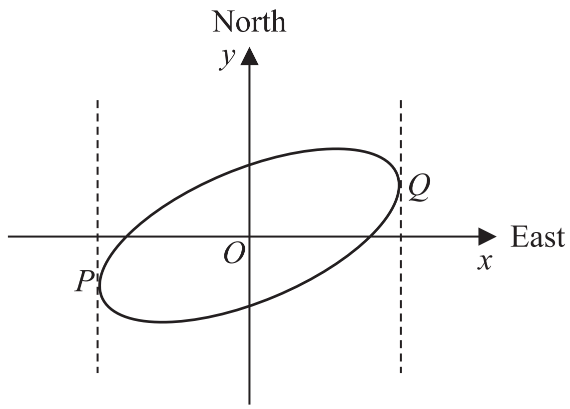


Figure 4

Figure 4 shows a sketch of the curve with equation $x^2 - 2xy + 3y^2 = 50$

↳ Implicitly.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y - x}{3y - x}$

(4)

The curve is used to model the shape of a cycle track with both x and y measured in km.

The points P and Q represent points that are furthest west and furthest east of the origin O, as shown in Figure 4.

Using part (a),

(b) find the exact coordinates of the point P.

(5)

(c) Explain briefly how to find the coordinates of the point that is furthest north of the origin O. (You **do not** need to carry out this calculation).

(1)

a)

$x^2 - 2xy + 3y^2 = 50$

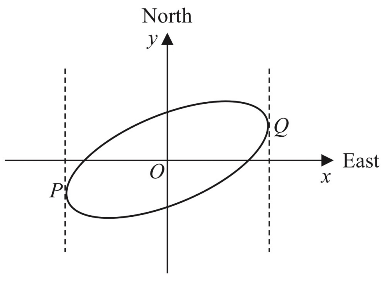


Figure 4

$2x - \frac{d}{dx}(2xy) + \frac{d}{dx}(3y^2) = 0$

$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \times \frac{dy}{dx}$

$2x - \frac{d}{dx}(2xy) + 6y \times \frac{dy}{dx} = 0$

$\frac{d}{dx}(u(x)\,v(y)) = u(x)\,v'(y) + v(y)\,u'(x)$

$u(x) = 2x \qquad v(y) = y$

$u'(x) = 2 \qquad v'(y) = \frac{dy}{dx}$

$+$

$2x - \left(2x \times \frac{dy}{dx} + 2y\right) + 6y\frac{dy}{dx} = 0$

$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(6y - 2x) = 2y - 2x$

$\frac{dy}{dx} = \frac{2y - 2x}{6y - 2x} \qquad \therefore \frac{dy}{dx} = \frac{y - x}{3y - x}$ as required.

b)

$\frac{dy}{dx} = \frac{y-x}{3y-x}$ $\quad x^2 - 2xy + 3y^2 = 50$

$\quad\quad 3y \quad\quad 3y$

At P and Q, $\frac{dy}{dx} \to \infty$ $\therefore 3y - x = 0$

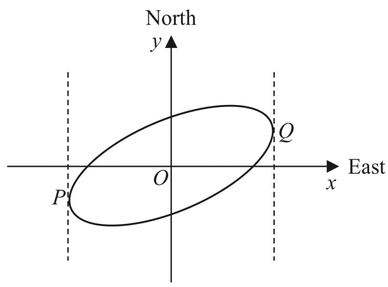


Figure 4

$\frac{a}{b} \to \infty$, $\quad b = 0$

$3y - x = 0 \quad \therefore \quad x = 3y \quad \Rightarrow \quad (3y)^2 - 2(3y)(y) + 3y^2 = 50$

$$9y^2 - 6y^2 + 3y^2 = 50$$

$$6y^2 = 50$$

$$3y^2 = 25$$

$$y^2 = \frac{25}{3}$$

$$y = \pm \frac{5}{\sqrt{3}}, \quad y = \pm \frac{5\sqrt{3}}{3}$$

Since point P has a negative y value, $P_y = -\frac{5\sqrt{3}}{3}$

$x = 3y = 3 \times \left(-\frac{5\sqrt{3}}{3}\right) = -5\sqrt{3}$

$\therefore P = \left(-5\sqrt{3}, -\frac{5\sqrt{3}}{3}\right)$

c)

$\frac{dy}{dx} = 0 \Rightarrow \frac{y-x}{3y-x} = 0$, $y - x = 0$

$\downarrow$

$y = x$

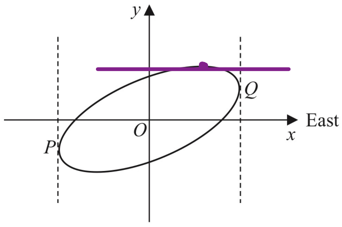


Figure 4

Solve $y = x$ and $x^2 - 2xy + 3y^2 = 50$ simultaneously, and choose positive solution.

2. The curve C has equation

$$x^2 \tan y = 9 \qquad 0 < y < \frac{\pi}{2}$$

(a) Show that

$$\frac{dy}{dx} = \frac{-18x}{x^4 + 81}$$

(4)

a) We want to use implicit differentiation to differentiate $x^2 \tan y = 9$

$x^2 \longrightarrow 2x$

$\tan y \longrightarrow \sec^2 y \frac{dy}{dx}$

Product Rule

$h(x) = f(x) \cdot g(x)$ then

$h'(x) = f'(x)\, g(x) + f(x)\, g'(x)$

$\Rightarrow \quad 2x \cdot \tan y + x^2 \sec^2 y \frac{dy}{dx} = 0$ (2)

1 for attempting to differentiate

1 for correct differentiation

We will use the trig identity : $\sec^2 y = 1 + \tan^2 y$ and $\tan y = \frac{9}{x^2}$

$\Rightarrow \tan^2 y = \frac{81}{x^4}$

$\Rightarrow \quad 2x \cdot \frac{9}{x^2} + x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{18}{x} + x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = 0$

$\Rightarrow \quad x^2\left(1 + \frac{81}{x^4}\right)\frac{dy}{dx} = -\frac{18}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{18}{x}}{x^2\left(1 + \frac{81}{x^4}\right)}$ (1) $= \frac{-18}{x^3\left(1 + \frac{81}{x^4}\right)}$ *

* $x^3\left(1 + \frac{81}{x^4}\right) = x^3\left(\frac{x^4 + 81}{x^4}\right) = \frac{x^4 + 81}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-18}{\frac{x^4+81}{x}}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{-18x}{x^4 + 81}$ as required. (1)

(b) Prove that C has a point of inflection at $x = \sqrt[4]{27} = (27)^{1/4}$

(3)

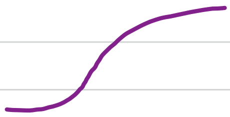
Quotient Rule :

$f(x) = \dfrac{h(x)}{g(x)}$ then

$f'(x) = \dfrac{h'(x) \cdot g(x) - h(x) \cdot g'(x)}{(g(x))^2}$

b) Part a : $\dfrac{dy}{dx} = \dfrac{-18x}{x^4+81}$

Point of inflection :



$-18x \longrightarrow -18$

$x^4+81 \longrightarrow 4x^3$ (1)

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-18(x^4+81) - 4x^3(-18x)}{(x^4+81)^2}$

$= \dfrac{-18x^4 - 1458 + 72x^4}{(x^4+81)^2}$

$= \dfrac{54x^4 - 1458}{(x^4+81)^2} = \dfrac{54(x^4-27)}{(x^4+81)^2} = \dfrac{d^2y}{dx^2}$ (1)

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{54(x^4-27)}{(x^4+81)^2}$

• At $x = \sqrt[4]{27} \Rightarrow x^4 = 27 \Rightarrow$ we can substitute this into $\dfrac{d^2y}{dx^2}$

$\Rightarrow$ For $x^4 = 27$, $\dfrac{d^2y}{dx^2} = \dfrac{54(27-27)}{(27+81)^2} = 0$

$\Rightarrow$ For $x^4 > 27$, $\dfrac{d^2y}{dx^2} > 0$

$\Rightarrow$ For $x^4 < 27$, $\dfrac{d^2y}{dx^2} < 0$

$\Rightarrow$ From this we can conclude that there is a point of inflection at $x = \sqrt[4]{27}$. (1)

3. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{dy}{dx} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

1 separate terms:

a) $\frac{d}{dx}(3y^2) = 6y\frac{dy}{dx}$

$\frac{d}{dx}(qxy) = qx\frac{dy}{dx} + qy$ ← apply the product rule.

when $y = u(x)v(x)$, $\frac{dy}{dx} = \frac{du}{dx}v + \frac{dv}{dx}u$

$\frac{d}{dx}(px^3) = 3px^2$

1

$3px^2 + qx\frac{dy}{dx} + qy + 6y\frac{dy}{dx} = 0$

rearrange to make $\frac{dy}{dx}$ the subject

$(qx + 6y)\frac{dy}{dx} = -3px^2 - qy$ 1

$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$ 1

b) when $x = -1$ and $y = -4$:

$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$ 1

$-p + 4q + 48 = 26$

$4q - p = -22$ 1

use original curve to make first equation

Question continued

$19x + 26y + 123 = 0$

$26y = -19x - 123$

$y = -\frac{19}{26}x - \frac{123}{26}$ $\therefore m = -\frac{19}{26}$ (1) rearrange normal equation to find gradient

$\frac{dy}{dx} = m$ at $(-1, -4)$ gradients are equal.

$\frac{-3px^2 - qy}{qx + 6y} = -\frac{19}{26}$ (1)

$\frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)} = -\frac{19}{26}$ substitute in $(-1, -4)$

simplify to make second equation.

$57p - 102q = 624$ (2) (1)

solve (1) and (2) simultanously to give:

$p = 2, q = -5$ (1)

solve simultaneously (by hand or using a calculator)

4.

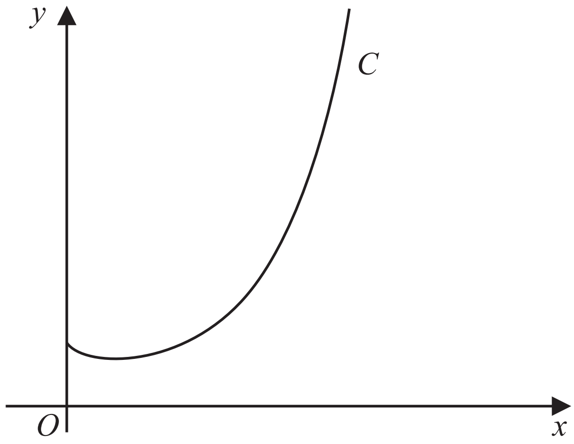


Figure 8

Figure 8 shows a sketch of the curve C with equation $y = x^x, \ x > 0$

(a) Find, by firstly taking logarithms, the x coordinate of the turning point of C.

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

(5)

The point $P(\alpha, 2)$ lies on C.

(b) Show that $1.5 < \alpha < 1.6$

(2)

A possible iteration formula that could be used in an attempt to find α is

$$x_{n+1} = 2x_n^{\,1-x_n}$$

Using this formula with $x_1 = 1.5$

(c) find x_4 to 3 decimal places,

(2)

(d) describe the long-term behaviour of x_n

(2)

a) $y = x^x$

$\ln y = \ln(x^x)$

$\Rightarrow \ln y = x \cdot \ln(x)$ 1

$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln(x) \cdot 1$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln(x)$ 2

$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{y} \cdot 0 = 1 + \ln x$

$\Rightarrow 0 = 1 + \ln x$

$\Rightarrow \ln(x) = -1$

$\Rightarrow e^{\ln x} = e^{-1} \Rightarrow x = \frac{1}{e} = 0.368$ 1

1

Turning Point ?

$\hookrightarrow \frac{dy}{dx} = 0$

log laws : $\ln(a^m) = m \cdot \ln(a)$

Product Rule : $h(x) = f(x) \cdot g(x)$

$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$x \rightarrow 1$

$\ln x \rightarrow \frac{1}{x}$

b) $y = x^x$

$x = 1.5 \Rightarrow y = 1.5^{1.5} = 1.84$
$x = 1.6 \Rightarrow y = 1.6^{1.6} = 2.12$ (1)

$P(\alpha, 2) \Rightarrow 1.84 < 2 < 2.12$, we also know that C is a continuous curve, hence $\underline{1.5 < \alpha < 1.6}$ (1)

c) $x_{n+1} = 2x_n^{1-x_n}$, $x_1 = 1.5$

$x_2 = 2 \cdot x_1^{1-x_1} = 2 \cdot (1.5)^{1-1.5} = 1.63299..$ (1)
$x_3 = 2 \cdot x_2^{1-x_2} = 1.46626...$
$x_4 = 2 \cdot x_3^{1-x_3} = 1.6731... \Rightarrow x_4 = \underline{1.673}$ (1)

d) $n \to \infty$, what happens to x_n?

- x_n fluctuates between 1 and 2 (1) 1, 2, 1, 2, ...
- x_n will be periodic with period 2 (1)